EFFECT OF NONSTATIONARITY IN THE MOTION OF AN OPEN FLUID FLOW

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ABSTRACT: A way of taking the effects of unsteady motion into consideration in boundary layer theory has been found recently [1]. In the meantime, in hydraulics ever-increasing interest is being shown in the application of the ideas and methods of boundary layer theory to the study channel flows. We have attempted here to apply this hydrodynamic approach in deriving the law of resistance for turbulent unsteady flows in open channels.

§1. DERIVATION OF THE SYSTEM OF EQUATIONS

We shall consider plane nonstationary flows of a viscous incompressible fluid described by the Navier-Stokes system of equations. We carry out the well-known transformation

$$t = (U / X)t_0, \quad x = Xx_0, \quad y = Yy_0,$$
$$u = Uu_0, \quad v = Vv_0, \quad p = \rho U^2 p_0.$$

Then this system of equations takes the dimensionless form

$$\frac{\partial u_0}{\partial t_0} + u_0\frac{\partial u_0}{\partial x_0} + \frac{XV}{UY} v_0 \frac{\partial u_0}{\partial y_0} =$$
$$= \frac{F_x X}{U^2} - \frac{\partial p_0}{\partial x_0} + \frac{\nu X}{Y^2 U}\left(\frac{Y^2}{X^2}\frac{\partial^2 u_0}{\partial x_0^2} + \frac{\partial^2 u_0}{\partial y_0^2}\right),$$
$$\frac{\partial v_0}{\partial t_0} + u_0\frac{\partial v_0}{\partial x_0} + \frac{XV}{UY} v_0 \frac{\partial v_0}{\partial y_0} =$$
$$= \frac{F_y X}{UV} - \frac{UX}{VY}\frac{\partial p_0}{\partial y_0} + \frac{\nu X}{Y^2 U}\left(\frac{Y^2}{X^2}\frac{\partial^2 v_0}{\partial x_0^2} + \frac{\partial^2 v_0}{\partial y_0^2}\right),$$
$$\frac{\partial u_0}{\partial x_0} + \frac{XV}{YU}\frac{\partial v_0}{\partial y_0} = 0. \tag{1.1}$$

Here we have taken the notation usually employed in hydrodynamics: t is time; (x, y) the Cartesian coordinate system, the x axis is directed along a fixed rectilinear contour; u, v are the respective velocity components along the x and y axes; p is the pressure; ρ and ν are the density and the kinematic viscosity coefficient of the fluid, respectively; F_x, F_y are the components of the body force; and X, Y, U, V are the scales of lengths and velocity components.

We impose a first constraint on the flow. The Reynolds number $R = UX/\nu$ is large, or more precisely, we can neglect quantities of an order of smallness O(1/R) and higher.

There are four arbitrary quantities U, V, X, and Y. In order to obtain a system of equations depending on a single parameter from (1.1), we subject these quantities to two variations of three conditions:

$$R = \frac{UX}{\nu}, \qquad \frac{XV}{YU} = 1, \qquad \nu X = Y^2 U, \tag{1.2}$$

$$R = \frac{UX}{\nu}, \qquad \frac{XV}{YU} = 1, \qquad X = Y. \tag{1.3}$$

As is known, conditions (1.2) lead the system of equations to a system of equations depending on the Reynolds number R in the vicinity of the boundary layer, and the conditions (1.3) in the region of the external flow.

In system (1.1), we shall neglect the quantities which are of the order of smallness O(1/R) or more, both under conditions (1.2) and (1.3).

Further, we impose a second constraint $g \gg dv/dt$, where g is the acceleration of gravity. This constraint yields an approximation of shallow-water theory. Shallow-water theory follows from the assumption that the component of the acceleration of a particle along the y-axis has an insignificant influence on the pressure [2]. Then, in the case of a heavy fluid and an inclined rectilinear bottom, according to system (1.1), we can write the system of equations in dimensional form:

$$\frac{du}{dt} = g\sin\alpha_0 - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2},$$
$$g\cos\alpha_0 + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{1.4}$$

Here α_0 is the acute angle between the y axis and the direction of the force of gravity.

Let the free surface be given by the expression $y = h(x, t)$. From the second equation of system (1.4) and the condition of constancy of the pressure on the free surface, we obtain

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = g\cos\alpha_0\frac{\partial h}{\partial x}.$$

Then, (1.4) takes the form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g\left(\sin\alpha_0 - \cos\alpha_0\frac{\partial h}{\partial x}\right) + \nu\frac{\partial^2 u}{\partial y^2}$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

Thus, the system of equations for the flow as a whole, under these constraints, has the form of a system of equations of boundary layer theory.

§2. TURBULENT FLOW

The equations of a turbulent boundary layer are known [1, 3]. It is not difficult to write them for a heavy fluid and an inclined bottom.

It was established in §1 that there is a definite relationship between the forms of the systems of equations of the boundary layer and the flow as a whole, namely: the system of equations describing the motion of a fluid over the region as a whole has the form of a system of equations of the boundary layer. We shall assume that this property of invariance of the forms of the system of equations of laminar boundary layer theory is also satisfied in the case of a turbulent flow. Then, for a turbulent open flow, we have the equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g\left(\sin\alpha_0 - \cos\alpha_0\frac{\partial h}{\partial x}\right) + \frac{1}{\rho}\frac{\partial \tau}{\partial y}$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{2.1}$$

Here τ is the friction stress.

The boundary conditions on the free surface are

$$\partial h / \partial t + u^\circ \partial h / \partial x = v^\circ \text{ (kinematic)}, \quad \tau = 0 \text{ (dynamic)}. \tag{2.2}$$

Here $u^\circ = u(t, x, h)$, $\qquad v^\circ = v(t, x, h)$.

The boundary conditions on the fixed boundary (on the bottom) are

$$u = 0, \quad \tau = 0 \quad \text{when } y = 0. \qquad (2.3)$$

§3. LAW OF RESISTANCE FOR UNSTEADY MOTION OF AN OPEN FLOW

We shall represent the ratio of the friction stress to the friction stress on the fixed boundary (on the bottom) in the form of a polynomial:

$$\frac{\tau}{\tau_0} = \sum_{i=0}^{n} b_i(t, x)\,\eta^i \qquad \left(\eta = \frac{y}{h}\right).$$

Here the coefficients b_i are determined from (2.1)–(2.3). We shall limit ourselves to the first three terms of the polynomial. We determine the coefficients b_0, b_1, and b_2 from the following conditions.

On the bottom

$$\frac{\tau}{\tau_0} = 1, \quad \frac{1}{\rho}\frac{\partial\tau}{\partial y} = g\left(\cos\alpha_0 \frac{\partial h}{\partial x} - \sin\alpha_0\right) \quad \text{when } \eta = 0.$$

This yields

$$\frac{\partial}{\partial\eta}\frac{\tau}{\tau_0} = \rho g \frac{h}{\tau_0}\left(\cos\alpha_0 \frac{\partial h}{\partial x} - \sin\alpha_0\right) = A.$$

On the free surface

$$\tau / \tau_0 = 0 \text{ when } \eta = 1.$$

Then we have

$$\tau / \tau_0 = 1 + A\eta - (1 + A)\eta^2 \qquad (0 \leqslant \eta \leqslant 1). \qquad (3.1)$$

On the other hand, the friction stress for a turbulent flow can be written in the form

$$\tau / \rho = \varepsilon\, \partial u / \partial y.$$

Thus, we obtain the differential equation for the velocity

$$\rho\varepsilon \partial u / \partial\eta = \tau_0 h\,[1 + A\eta - (1 + A)\eta^2]. \qquad (3.2)$$

For small values of y the Al'tshul-Hinze formula [4, 5] is valid, i.e.,

$$\varepsilon = \alpha u_* y, \qquad u_* = \sqrt{|\tau_0|/\rho}. \qquad (3.3)$$

Here α is the universal constant, u_* is the dynamic velocity. We take $\varepsilon = \alpha u_* f(y)$ for any y. We shall seek the function $f(y)$, following the Satkevich method, so that in case of uniform motion we obtain the logarithmic velocity profile, which corresponds better to the hydrometric data than others which have been proposed. Then it is not difficult to obtain

$$f(y) = y(1 - \eta). \qquad (3.4)$$

We note that with uniform motion $A = -1$. Taking $\tau_0/\rho = u_*^2 \cdot \operatorname{sign} w$ and (3.4) into consideration, we obtain from (3.2)

$$\frac{u}{u_* \operatorname{sign} w} = \frac{1}{\alpha}[\ln\eta + (1 + A)\eta] + C(t, x), \quad w = \int_0^1 u\, d\eta. \qquad (3.5)$$

Here C(t, x) is an arbitrary function. The function C(t, x) is determined from the condition that $u = \beta u_*$ when $\eta = k/h$, where k is the average height of the roughness effect, and β is the universal constant [4]. Integrating (3.5) with respect to η from 0 to 1 and assuming that $k/h \ll 1/2$, we obtain

$$\frac{w}{u_* \operatorname{sign} w} = \frac{1}{\alpha}\left[\ln\frac{h}{k} + \frac{A}{2} + \alpha\beta - \frac{1}{2}\right].$$

From this, determining $\lambda \geq 0$ from the relation $\tau_0/\rho = \lambda\,|w|\,w$, we obtain

$$\sqrt{\lambda} = \frac{1 + (1 + f_1 f_2)^{1/2}}{2\alpha^{-1} f_2},$$

$$f_1 = \frac{2}{\alpha^2}\frac{gh}{|w|\,w}\left(\sin\alpha_0 - \cos\alpha_0\frac{\partial h}{\partial x}\right), \quad f_2 = \ln\frac{h}{k} + \alpha\beta - \frac{1}{2}.$$

If we take the linear relation (3.3), which is valid, generally speaking, for small y, as the function $f(y)$, we shall also take account of molecular viscosity, that is, we represent the stress in the form [4]

$$\tau / \rho = (\nu + \varepsilon)\partial u / \partial y. \qquad (3.6)$$

In this case, taking $\alpha u_* h/\nu \gg A$, $k/h \ll 1$, we obtain the expression for $\sqrt{\lambda}$ in the following form:

$$\sqrt{\lambda} = \frac{1 + [1 + {}^2/_3 f_1 (f_3 - {}^1/_6)]^{1/2}}{2\alpha^{-1}(f_3 - {}^1/_6)},$$

$$f_3 = \ln\frac{\alpha\nu^{-1}\sqrt{\lambda}\,|w|\,h}{1 + \alpha\nu\sqrt{\lambda}\,|w|\,k} + \alpha\beta - 1.$$

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